**Ornstein-Uhlenbeck Process**

So the Wiener process forms the backbone of the Ornstein-Uhlenbeck process, which seems to adequately describe all the transfer matrix modeling going on.

**Scalar OU Process**

Now let’s say that W(t) drives the evolution of another random variable X(t) such that:



Now a(X,t)dt is the ‘drift’ term, whereas bdW(t) is a random white noise ‘variance’ term. What does this mean? Well this gives an equation for each increment’s probability distribution. What is the probability distribution of dX? Well in the Ito version, it’s linearly related to dW which is Gaussian distributed. Probability distribution of dXj is just the same as probability distribution of a(Xj)dt + b(Xj)dW(tj). So it’s just a normal distribution as well, with a mean of a(X)dt and variance b(X)2Ddt. Note that the p.d.f. of dW(t) is independent of X’s though, at that moment. The p.d.f. of each dXj would be different. Don’t think we can say what the pdf of dX is for general values of λ though, because dW and X(tm+λ) would be correlated? We can get the pdf of X(t) via:



which translates to the following discrete version:



Note X is a Markov process because Xn+1 only depends on Xn (but maybe not a Markov process if λ ≠ 0). But it may not be a Martingale since <dX> = a(X)dt ≠ 0 probably. Even though the pdf of each dX is Gaussian, each increment has a different mean and variance, usually, and so the central limit theorem doesn’t apply and so p.d.f. of X(t) won’t usually be Gaussian. We’ll find it useful to switch from an Ito OU process to a Stratonovich OU process and vice versa. Let’s do Ito → Stratonovich. This will mean starting with an ODE where a(X(tj+0)) and b(X(tj+0)) and switching to a(X(tj+1/2)) and b(X(tj+1/2)). Preview: making the switch in the a term will cost nothing to O(dt), but we will gain an extra term doing so in the b term. Note we’ll say dt = ti+1 – ti and dW = W(ti+1) – W(ti) for short.



Note we kept only terms up to O(dt). Filling in the ΔX term,



Now we can replace the b(X(ti+1/4),ti+1/4) with b(X(ti+1/2),ti+1/2) since the error involved in doing so is O(dW), which will ultimately vanish in the limit. So then proceeding:



Now we can as usual replace the dW product with its average, which is Ddt/2. So we get:



Dividing by dt, we get:



which is our result. So in general, going from one to the other we have:



And,



**Example**

So let’s say I have an Ito process:



What is the Stratonovich version?



**Example**

What if we make a change of variables in:



Consider: X = Y2. Then,



Is this true? Say we go to the Stratonovich version first,



and then make the change of variables,



and then convert back to Ito,



So the two procedures are not equivalent. We have to do the Stratonovich version first, for this to be valid. If we borrow result for Stochastic derivative below, of F(X),



then we can bolster this supposition.



and when going to the Ito version, we have:



And *this* matches.

**Example**

Say we have:



If Y = X2, what is dY?



**Integral functions of OU processes**

Let’s consider a function f(X,t). We want to examine the integral of this quantity, defined as:



To that end, let’s consider the antiderivative F(X,t) for which ∂F/∂X = f. Then,



Now we can relate ΔX to Δt, presumably as we’ve done before:



But we should check:



So yeah this should check out again. Then we can stop at this order in the expansion, and so then we have our result, backing up a little:



And we could solve for the integral too. Denoting ∂F/∂X = f.



**Example**

Suppose an OU process is given by:



What is X(t)? Well, let f(X) = 1. Then,



Okay that didn’t help.

**Example**

Suppose an OU process is given by:



What is ∫X(t)dX? Well, let f(X) = X. Then,



Looks like we can do the same class of integrals as with the Wiener process.

**Differential functions of OU process**

Now we’ll reverse the integral formula and consider differentiation. We’d define:



and so,



So dividing by dt, we finally obtain:



Or more concisely,



So if we do the ‘extended calculus’ with Taylor series, then it works out. Now let’s consider the product rule:



and so we have:



Again, let’s consider the Ito heuristic:



So using the extended product rule would deliver the Ito version, if D were symmetric. I’d expect it always is. What of the chain rule?



And so we have:

